REPRESENTATION OF SOME FINITE CONNECTIVE SPACES BY GRAPHS

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Abstract: Connective spaces generalize the concept of connectedness. In this paper, Some facts and concepts about the connective spaces including different connective substructures is presented. A method of representation of finite connective spaces by simple graphs are described.

Keywords: Connective space, c-space, Graph, Connectedness:

Introduction

The concept of connectivity is very important in the analysis, this motivated **'G. Matheron & J. Serra'** (in 1988) to propose an axiomatic approach to connectivity. Their approach is based on the observation that the most standard notions of connectivity share the properties that the empty set and the singletons of the underlying space are connected, and that unions of connected objects are connected. These properties may be considered to be a minimal set of desirable requirements for connectivity. This work also was pursued by **'Christian Ronse'**, **S. Dugowson** and others. In 2006, **Muscat & Buhagiar** introduced a special case (satisfying some additional properties) of the spaces studied by **'G. Matheron and J. Serra'** which called connective spaces. A connective space is a pair $(X, \mathcal{C})$ of a non-empty set X and a collection $\mathcal{C}$ of subsets of X with a certain properties. This paper presented idea of representation of finite connective spaces by simple directed acyclic graphs which so called the generic graphs also it contained some basic results for our further work.

Many conventional high-voltage power supply designs can be used to produce the short duration voltage pulse to form the plasma discharge [6, 7].

Temporal jitter in the induced current pulse arises from the stochastic nature of the avalanche ionisation process and is crucially dependent on the rise time of the voltage pulse, which should be minimised. It is also desirable characteristic of to minimise applied voltage for full ionisation to reduce electrical noise in the experimental environment. The goal here is to investigate the hydrogen-filled capillary discharge waveguide for the use of laser wakefield acceleration.

2 Experimental Setup

The work presented here was carried out in the TOPS laser laboratory of the University of Strathclyde. It involved testing of the waveguides to analyse their plasma discharge characteristics, which can be used in experiments such as laser wakefield acceleration [8,9].

The Theory of Methods

Definition 1.[2] A *connective space* $(X,\mathcal{C})$ is a set X together with a collection of subsets $\mathcal{C}$, such that the following axioms hold:

$(i)\forall\ C \subseteq \mathcal{C}, \bigcap C \neq \emptyset \Longrightarrow \bigcup C \in \mathcal{C}$.

$(ii)\forall\ x \in X \Longrightarrow \{x\} \in \mathcal{C}$.

(iii) Given any non-empty sets A , $B \in \mathcal{C}$ with $A \cup B \in \mathcal{C}$,then $\exists\ x \in A \cup B$ such that $\{x\} \cup A \in \mathcal{C}$ and $\{x\} \cup B \in \mathcal{C}$.

(iv) If $A, B, C_i \in \mathcal{C}$ are disjoint, $\forall i \in I$ and $A \cup B \cup \bigcup_{i \in I} C_i \in \mathcal{C}$, then $\exists J \subseteq I$ such that $A \cup \bigcup_{j \in J} C_j \in \mathcal{C}$ and $B \cup \bigcup_{i \in I-J} C_i \in \mathcal{C}$.

The set X is called *Carrier* of the space $(X, \mathcal{C})$, and the collection $\mathcal{C}$ is called the *connective structure* or *connectology* of X, and its elements are called the *connected subsets* of X .

A connective space is called *finite* if its carrier is a finite set.

Remark 2

(1) In the previous definition, we can add the axiom that the empty set is connected although this follows from (i) .

(2) In connective spaces, the connected sets with two elements are called edges, and the spaces that satisfy (i) and (ii) only, will be called *c-spaces* and the corresponding $\mathcal{C}$ a c-structure.

Example 1

1. Topological spaces with the connected sets are connective spaces .

2. The real line $\mathbb{R}$ together with all intervals and singletons is a connective space, and is called the *real connective space.*

3. Let X be any set, the collection $\mathcal{D} = \{\emptyset, \{x\}: x \in X\}$ is a connective structure on X and $(X, \mathcal{D})$ is called the *discrete connective space*. The collection $\mathcal{J} = \mathcal{P}(X)$ is also a connective structure on X and $(X, \mathcal{J})$ is called *the indiscrete connective space*.

Remark 3 Connective structures of a set X are partially ordered , since if $\boldsymbol{\mathcal{C}}^*$ is the family of connective structures of X, $\mathcal{C}_1, \mathcal{C}_2 \in \boldsymbol{\mathcal{C}}^*$ then $\mathcal{C}_1 \leq \mathcal{C}_2 \Leftrightarrow \mathcal{C}_1 \subseteq \mathcal{C}_2$, so $\mathcal{C}_2$ is called *coarser* (*weaker*) than $\mathcal{C}_1$ and $\mathcal{C}_1$ is called *finer* (*stronger*) than $\mathcal{C}_2$. The weakest Connective structure is the indiscrete structure on X and the strongest Connective structure on X is the discrete structure.

Definition 4.[8] Let X be a non-empty set, and $\mathcal{B}$ is a collection of subsets of X. The strongest connective structure on X which contains $\mathcal{B}$ is called the *connective structure generated* by $\mathcal{B}$, and it is denoted by $[\![\mathcal{B}]\!]$ Thus

$$[\![\mathcal{B}]\!] = \bigcap\{\mathcal{C} : \mathcal{B} \subset \mathcal{C}\}$$

and $\mathcal{B}$ is called a *basis* for the connective structure $[\![\mathcal{B}]\!]$.

Definition 5.[8] Let $(X, \mathcal{C})$ be a connective space . A connected subset K of X is called *reducible* if it belongs to the connective structure generated by others, that is

$$K \in [\![\, \mathcal{C} \setminus \{K\} \,]\!]$$

A non-empty connected subset of X is said to be *irreducible* if it is not reducible

Example 2 The singletons are irreducible sets in any connective space .

Definition 6 A space $(X, \mathcal{C})$ is said to be *irreducible* if X is an irreducible connected subset of itself . It is said to be *distinguished* if each of its non-empty connected subsets is irreducible .

Example 3 A discrete connective space is distinguished space .

Theorem 1 A connective structure on a given finite set is characterized by the set of irreducible connected subsets, which is the minimal set of subsets which generates this structure .

Proof Let $(X , \mathcal{C})$ be a connective space, and $I(X)$ denote the set of all irreducible connected subsets of X. Then for any $\mathcal{A} \subseteq \mathcal{P}(X)$ such that $[\![\mathcal{A}]\!] = \mathcal{C}$, one has $I(X) \subseteq \mathcal{A}$, since each set $C \in [\![\mathcal{A}]\!]$ which is not in $\mathcal{A}$ is reducible, hence $[\![I(X)]\!] \subseteq [\![\mathcal{A}]\!]$. On the other hand, suppose that K is reducible connected subset of X, such that $K \in [\![\mathcal{A}]\!]$, $K \notin \mathcal{A}$,and since a reducible set belongs to the structure generated by others, so $K \in [\![I(X)]\!]$, thus $[\![\mathcal{A}]\!] \subseteq [\![I(X)]\!]$ consequently $[\![I(X)]\!] = [\![\mathcal{A}]\!] = \mathcal{C}$.

For a proof that $I(X)$ is the minimal set of subsets which generates $\mathcal{C}$, suppose that there is a set of irreducible connected subsets $I^*(X)$ such that $\mathcal{C} = [\![I^*(X)]\!]$ and $I^*(X) \subseteq I(X)$, so if $K \in I(X)$ is an irreducible connected set, then $K \in [\![I(X)]\!] = \mathcal{C} = [\![I^*(X)]\!]$, hence $K \in I^*(X)$, because if $K \notin I^*(X)$ then K is a reducible. This implies that $I^*(X) = I(X)$ and it is a minimal.

Theorem 2 Let $(X,\mathcal{C})$ be a finite connective space, a subset K of X is a reducible if there are two connected sets $A \subsetneq K$ and $B \subsetneq K$ such that

$K = A \cup B$ and $A \cap B \neq \emptyset$

Proof Let K be a reducible connected subset of X, then $K \in [\![\mathcal{C} \setminus \{K\}]\!]$, so there are two connected sets C_1 and C_2 which belong to $\mathcal{C}$, such that $K = C_1 \cup C_2$ and $C_1 \subsetneq K$, $C_2 \subsetneq K$. Now from axiom (iii) of definition(1), there exists $x \in K$ such that $\{x\} \cup C_1 \in \mathcal{C}$ and $\{x\} \cup C_2 \in \mathcal{C}$.

Choose $A = \{x\} \cup C_1$ and $B = \{x\} \cup C_1$, which completes the proof .

Definition 7 Let X be a finite connective space . A *generic point* of X is a non-empty irreducible connected subset of X. The *generic graph* G_X of X is the directed graph whose vertices are the generic points of X, and $a \longrightarrow b$ is a directed edge of G_X if and only if $a \supsetneq b$ and there is no generic points c such that $a \supsetneq c \supsetneq b$.

Theorem 3 A finite connective space X is characterized by its generic graph G_X.

Proof Let $(X, \mathcal{C})$ be a finite connective space, every singleton is an irreducible connected subset, then it is a generic point in G_X and $X = \bigcup_{a \in G_X}\{a\}$, but (from theorem 1) any finite connective structure is characterized by the set of irreducible connected subsets, which is the minimal set of subsets which generates this structure , so $\mathcal{C} = [\![a : a \in G_X]\!]$.

Now if K_1 , K_2 are irreducible connected subsets in X such that $K_1 \supsetneq K_2$ and there is no K^* such that $K_1 \supsetneq K^* \supsetneq K_2$, but K_1 , K^* and K_2 are generic points in G_X, hence it is clearly that (K_1 , K_2) is a directed edge in G_X. On the other hand, let (a, b) be a directed edge in G_X, so $a \supset b$ are both irreducible connected subsets of X, and if there exists a generic point c such that $a \supset c \supset b$ then $a = c \cup b$ and a is reducible, contradicting the irreducibility of a

Generally, if the connective space has a generic graph then it is called graphical

Remark 8 The singleton appears as a sink in G_X i.e. a vertex with no outgoing edges.

Note that not every finite acyclic directed graph is G_X. For example, the directed acyclic directed graph $a \rightarrow b$ is not a G_X. Because since b is a proper subset of a, then a belongs to at least one element x which different of b, but there is no vertex x that represents the singleton of x .

Example 4 Let $X = \mathrm{B}_3$ (Borromean space of three points), such that $X = \{$ blue, green, red $\}$ and $\mathcal{C} = \{\emptyset$, {blue}, {green}, {red}, X$\}$, as in the following figure ;

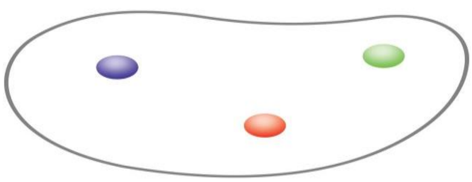


Figure 1

Then $(X, \mathcal{C})$ has four generic points which identified with the three points in space, the fourth, which identified to the whole space . Its generic graph is represented by the next figure;

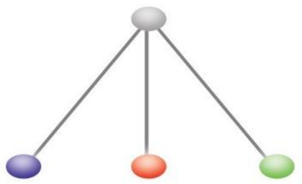


Figure 2 . Generic graph of the space in figure 1

Example 5 The connective space $(X, \mathcal{C})$ defined by $X =$ { orange, bud green, olive green, mauve, red, pink, green, blue, sky blue } and $\mathcal{C} =$ {$\emptyset$, {orange}, {bud green}, {olive green}, {mauve}, {red},{pink},{green}, {blue}, {sky blue}, {orange, bud green, olive green}, {mauve, red, pink}, {green, blue, sky blue}, X}.
As in the following figure ;

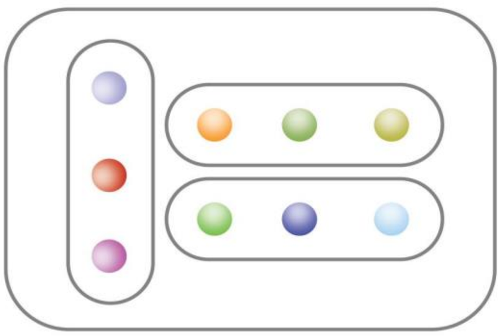


Figure 3

It is clear that this space has 13 generic points and it is generic graph is directed tree as the following ;

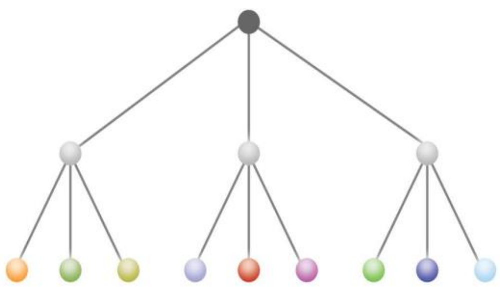


Figure 4 . Generic graph of the space in figure 3

Example 6 Let A_5 ={green, pink, orange, blue, sky blue} , and $\mathcal{A}_5$ ={{green}, {pink}, {orange}, {blue}, {sky blue}, {green, pink}, {green, pink, orange}, {green, pink, orange, blue}, X }.
The space is represented by the following figure

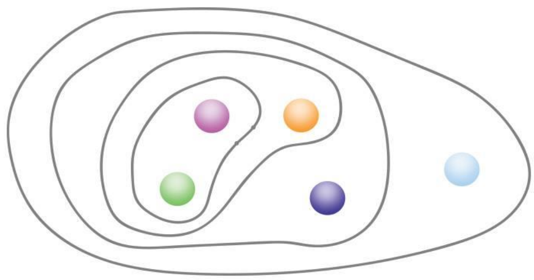
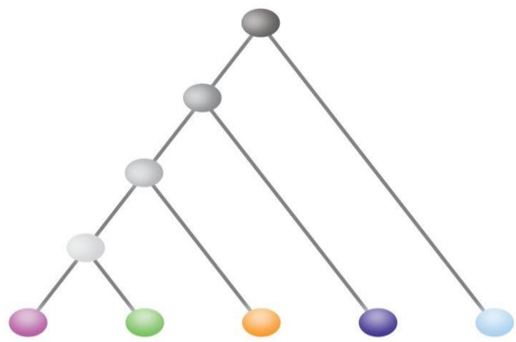


Figure 5

Then the connective space $(A_5, \mathcal{A}_5)$ has nine generic points, and its generic graph is represented by the following figure ;

Figure 6 . Generic graph of the space in figure 5



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Remark 9 The order of a finite connective space is the maximal length of paths in its generic graph . In the previous example, the connective space is of order five, while in the example (5), a space is of the fourth order**.**
iii) X is irreducible iff G_X has exactly one source, i.e. a vertex with no incoming edges.
(iv) X is distinguished iff there is no triple (a, b, c) of distinct vertices in G_X such that $(a \to b)$ and $(b \leftarrow c)$ are in G_X.
(v) X is connected and distinguished iff G_X is a directed tree .

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